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# On the bounds for the curvature and higher derivatives of the Isgur-Wise function

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## Abstract

We discuss constraints imposed on the zero-recoil curvature and higher derivatives of the Isgur-Wise function by a general quark model. These constraints are expressed as bounds for a given slope parameter, and compared with those based upon analyticity properties of QCD spectral functions. Our results also indicate that in the analysis of the experimental data for semileptonic  $B \rightarrow D^{(*)}$  decays it may be important to include at least the third term in the form factor expansion about the zero recoil point.

# 1 Introduction

It has been widely recognized for some time that heavy quark symmetry (HQS) [1, 2] enormously simplifies the analysis of the semileptonic  $B \rightarrow D^{(*)}$  decays. The six form factors needed for the description of these decays are in the heavy quark limit reduced to a single unknown form factor, the Isgur-Wise (IW) function  $\xi(\omega)$ , where  $\omega = v_B \cdot v_{D^{(*)}}$  is the product of the four-velocities of the two mesons. Furthermore, HQS also provides us with a prediction for the normalization of the universal form factor at the zero-recoil point, i.e.  $\xi(1) = 1$ . As a consequence of that, normalizations of the physical form factors  $\mathcal{G}(\omega)$  (for  $B \rightarrow D$  decay) and  $\mathcal{F}(\omega)$  (for  $B \rightarrow D^*$  decay),<sup>1</sup> are determined up to radiative and power corrections. Therefore, by extrapolating the experimental data for the differential decay rate to  $\omega = 1$  one can obtain an accurate measurement of the Cabibbo-Kobayashi-Maskawa parameter  $|V_{cb}|$ . The decay  $B \rightarrow D^* l \bar{\nu}_l$  is ideally suited for this purpose [3]. It is experimentally clean mode, and the decay rate at zero recoil is protected by Luke's theorem against first order  $1/m_Q$  corrections [4].

This analysis has already been performed by several experimental groups [5]-[8]. In [5] and [6] the fit to the data assumed a linear form for  $\mathcal{F}(\omega)$ , i.e.  $\mathcal{F}(\omega) = \mathcal{F}(1)[1 - \hat{a}(\omega - 1)]$ ,<sup>2</sup> while in [7] and [8] fits with quadratic form of the hadronic form factor were also attempted, but with the conclusion that with the existing data samples it was not possible to distinguish between the linear parametrization and those with more degrees of freedom. It should be obvious that retaining only the first term in the expansion will inevitably lead to an underestimate of the slope parameter. This point was made some time ago by Burdman [9], who included the curvature (quadratic) term in a two parameter

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<sup>1</sup>In the absence of symmetry breaking corrections these form factors would coincide with the IW function  $\xi(\omega)$ .

<sup>2</sup>In [5] several other parametrizations for  $\mathcal{F}(\omega)$  were also used, but all with only one degree of freedom.

analysis. Because of the statistical uncertainty introduced in a two parameter fit, it is clearly important to obtain some theoretical insight about the expansion parameters, in order to guide the extrapolation to  $\omega = 1$ .

This issue has already been addressed in several papers [10]-[13], by employing analyticity properties of QCD spectral functions and unitarity. The resulting bounds proved to be weak due to the presence of the  $\Upsilon$  poles below the  $B\bar{B}$  threshold (or possible  $B_c$  states below the  $BD^{(*)}$  threshold in [13]). In the most recent work [14] Caprini and Neubert (CN) improved bounds for the zero-recoil slope and curvature (i.e. the second term in the expansion) of  $\mathcal{F}(\omega)$  and  $\mathcal{G}(\omega)$  by identifying a specific  $B \rightarrow D$  form factor which does not receive contributions from the ground state  $B_c$  poles. These authors have derived constraints between the slope and curvature of that form factor using analyticity properties of QCD spectral functions and unitarity, and then used heavy quark symmetry to relate these results to corresponding constraints for  $\mathcal{F}(\omega)$  and  $\mathcal{G}(\omega)$ .

In this letter we discuss an alternative approach for obtaining allowed regions for the curvature and higher derivatives of the IW function  $\xi(\omega)$  for a given value of the slope. Our results are obtained in the heavy quark limit and in the valence quark approximation, with some physical input about the shape of the heavy-light meson wave function. An advantage of the method is that, given the above assumptions, constraints can be obtained not only for the curvature, but also for any higher terms in the expansion. Even though we do not take symmetry breaking corrections into account, we believe that our results may also shed some light for guiding the experimental extrapolation to  $\omega = 1$ , especially for estimating the higher order terms in the form factor expansion.

The rest of the paper is organized as follows: in Section 2 we present the general valence quark model expressions for the IW function, and also for the particular terms in its expansion about  $\omega = 1$ . In Section 3 we show how to extract bounds on the higher expansion parameters, if the slope of the IW function is given. Results are discussed in

Section 4, and conclusions are presented in Section 5. In particular, we conclude that in the case of  $B \rightarrow D^{(*)}$  semileptonic decays an expansion of the IW function about the zero recoil point is converging slowly for  $\omega$  close to the maximum velocity transfer  $\omega_{max}$ .<sup>3</sup>

## 2 IW form factor in a general quark model

In the valence quark approximation the expression for the IW function describing the  $B \rightarrow D^{(*)}$  transitions, is given in terms of the  $S$ -wave radial wave function  $R(r)$  and energy  $E$  of the light degrees of freedom in the ground state heavy-light meson [15]-[19],

$$\xi(\omega) = \frac{2}{\omega + 1} \langle j_0(kr) \rangle , \quad (1)$$

where  $j_0$  is the spherical Bessel function and

$$k = 2E \sqrt{\frac{\omega - 1}{\omega + 1}} . \quad (2)$$

The expectation value  $\langle F(r) \rangle$  is given by

$$\langle F(r) \rangle = \int_0^\infty r^2 dr |R(r)|^2 F(r) . \quad (3)$$

We define expansion of  $\xi(\omega)$  around  $\omega = 1$  as

$$\xi(\omega) = 1 - a(\omega - 1) + b(\omega - 1)^2 - c(\omega - 1)^3 + d(\omega - 1)^4 + \dots . \quad (4)$$

Using (1) it is straightforward to find expressions for the slope  $a$  and higher order terms in (4) [17, 19]. We list here the first four terms:

$$a = \frac{1}{2} + \frac{1}{3} E^2 \langle r^2 \rangle , \quad (5)$$

$$b = \frac{1}{4} + \frac{1}{3} E^2 \langle r^2 \rangle + \frac{1}{30} E^4 \langle r^4 \rangle , \quad (6)$$

$$c = \frac{1}{8} + \frac{1}{4} E^2 \langle r^2 \rangle + \frac{1}{20} E^4 \langle r^4 \rangle + \frac{1}{630} E^6 \langle r^6 \rangle , \quad (7)$$

$$d = \frac{1}{16} + \frac{1}{6} E^2 \langle r^2 \rangle + \frac{1}{20} E^4 \langle r^4 \rangle + \frac{1}{315} E^6 \langle r^6 \rangle + \frac{1}{22680} E^8 \langle r^8 \rangle . \quad (8)$$

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<sup>3</sup>For  $B \rightarrow D^* l \bar{\nu}_l$   $\omega_{max} \simeq 1.5$ , and for  $B \rightarrow D l \bar{\nu}_l$  decays  $\omega_{max} \simeq 1.6$ .

Note that all quantities are positive definite. From (5) one finds  $E^2\langle r^2\rangle = 3(a - \frac{1}{2})$ , which can be used to reexpress (6), (7) and (8) in terms of  $a$  as

$$b = \frac{1}{4} + (a - \frac{1}{2}) + \frac{3}{10}\beta(a - \frac{1}{2})^2, \quad (9)$$

$$c = \frac{1}{8} + \frac{3}{4}(a - \frac{1}{2}) + \frac{9}{20}\beta(a - \frac{1}{2})^2 + \frac{3}{70}\gamma(a - \frac{1}{2})^3, \quad (10)$$

$$d = \frac{1}{16} + \frac{1}{2}(a - \frac{1}{2}) + \frac{9}{20}\beta(a - \frac{1}{2})^2 + \frac{3}{35}\gamma(a - \frac{1}{2})^3 + \frac{1}{280}\delta(a - \frac{1}{2})^4. \quad (11)$$

Here, we defined dimensionless quantities

$$\beta = \frac{\langle r^4 \rangle}{\langle r^2 \rangle^2}, \quad (12)$$

$$\gamma = \frac{\langle r^6 \rangle}{\langle r^2 \rangle^3}, \quad (13)$$

$$\delta = \frac{\langle r^8 \rangle}{\langle r^2 \rangle^4}. \quad (14)$$

From (5) one can see that  $a \geq 1/2$ , and therefore it is immediately evident that all of the above parameters must be positive.<sup>4</sup> However, we can bound them more stringently in order to yield more useful restrictions on the allowed ranges of  $b$ ,  $c$ , etc.

### 3 Bounds

Since all terms which accompany  $\beta$ ,  $\gamma$  and  $\delta$  in (9)-(11) are positive definite, it should be obvious that by restricting the allowed range for those parameters we also restrict the range of allowed values of  $b$ ,  $c$  and  $d$ , for a given slope parameter  $a$ . In other words, we

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<sup>4</sup>The bound  $a \geq 1/2$  is a consequence of the prefactor  $2/(\omega + 1)$  in (1), derivation of which is discussed in depth in [19], and which is closely related to the valence quark approximation. The relationship between the HQET sum rules and quark models was investigated in [20], where it was shown that Bjorken [21, 22] and Voloshin [23] sum rules can be used to construct a model which is self-consistent in the heavy-quark limit.

want to find  $\beta_{min}(\beta_{max})$  so that

$$\beta_{min} \leq \beta \leq \beta_{max} , \quad (15)$$

and similarly for other parameters.

Without making any further assumptions about the particular form of the heavy-light wave function, lower bounds for  $\beta$ ,  $\gamma$  and  $\delta$  can be obtained by considering the Schwartz-type inequality

$$\langle r^{2m}(r^{2n} - \langle r^2 \rangle^n)^2 \rangle \geq 0 . \quad (16)$$

For  $m = 0, 1$  and  $n = 1, 2$  this yields

$$\beta_{min} = 1 , \quad (17)$$

$$\gamma_{min} = 1 , \quad (18)$$

$$\delta_{min} = 1 . \quad (19)$$

In order to estimate the upper bounds we need some physical input. Let us for the moment assume that the ground state wave function of the light degrees of freedom in a heavy-light meson is given in the form

$$R(r) \propto \exp(-r^k) , \quad (20)$$

where  $k > 0$ . Note that any scale or normalization dependence in the wave function is unimportant, since it would cancel out in the ratios (12)-(14). For example,  $k = 2$  would correspond to the harmonic oscillator wave function, which is (with appropriate scale parameter) often used as an approximation for the meson wave function [24, 25]. Case  $k = 1$  corresponds to the pure exponential, which seems to be favored by lattice QCD [26]. Using (20) one can find the expression

$$\frac{\langle r^{2n} \rangle}{\langle r^2 \rangle^n} = \frac{\Gamma(\frac{2n+3}{k})[\Gamma(\frac{3}{k})]^{n-1}}{[\Gamma(\frac{5}{k})]^n} , \quad (21)$$

which with  $n = 2, 3$  and  $4$  yields  $\beta$ ,  $\gamma$  and  $\delta$  for any  $k$ . In Table 1 we list the actual numbers for several cases of interest. Clearly, as the value of  $k$  gets smaller, parameters  $\beta$ ,  $\gamma$ , and  $\delta$  get larger. Therefore, the smallest acceptable value of  $k$  will lead to the largest acceptable values for our parameters. Since lattice simulations [26] support a pure exponential falloff of the meson wave function ( $k = 1$ ), one might argue that choosing, for example,  $k_{min} = 1/2$  would leave more than enough room for possible uncertainties in the specific choice (20) of the long distance behavior of the wave function. In that case we would have (see Table 1)

$$\beta_{max} \approx 5.67 , \quad (22)$$

$$\gamma_{max} \approx 107.19 , \quad (23)$$

$$\delta_{max} \approx 5091.38 . \quad (24)$$

We wish to emphasize here that any value  $k_{min} < 1$  would be acceptable as far as this part of the analysis is concerned, because it is essentially guided only by the information obtained from the lattice [26]. By choosing  $k_{min}$  closer to 1, one would obtain quite narrow range for all parameters under consideration, as we shall see in the following section.

## 4 Results

Let us first discuss the second term in the expansion (4). In Figure 1 with full lines we show the acceptable range for the curvature  $b$  as a function of the slope  $a$ , with a particular choice of  $k_{min} = 1/2$ . The lower bound (denoted by *L.B.*) follows from (17), and the physically motivated result with  $k = 1$  is shown as well. With dashed lines we further show the result of the analysis performed in [14], which was also obtained in the heavy quark limit, but (unlike ours) includes short-distance corrections. To be completely

clear, we give here (in our notation) the form factor definition used in [14]

$$\tilde{\xi}(\omega) = \tilde{\xi}(1)[1 - \tilde{a}(\omega - 1) + \tilde{b}(\omega - 1)^2 + \dots] . \quad (25)$$

To avoid confusion, we have used tilde with the CN expansion parameters and form factor. When the short-distance corrections ( $\tilde{\xi}(1) \simeq 1.02$ ) are neglected, (25) coincides with (4).<sup>5</sup>

Since the CN ellipse shown in Figure 1 is rather narrow, these authors also give the approximate relation between the slope and the curvature as

$$\tilde{b} \simeq 0.72\tilde{a} - 0.09 . \quad (26)$$

On the other hand, our result with  $k = 1$ , yields

$$b \simeq -0.06 + 0.25a + 0.75a^2 , \quad (27)$$

which is within the CN bounds for values of  $a$  smaller than about 0.7, and grows faster than (26) with increasing  $a$ . From the Figure 1 one can see that given a value for the IW function slope  $a$ , the valence quark model yields a range for the curvature  $b$  which is comparable in size to the CN bounds, but with somewhat higher values for  $b$  when  $a$  is greater than about 0.7. We remind the reader that (5) requires  $a \geq 0.5$ .

The CN approach is expected to break down for higher than the second terms in the expansion of the IW form factor. The reason is the possible presence of the sub-threshold singularities which are due to scalar  $B_c$  resonances, or due to states of the form  $(B_c^{(*)} + h)$ , where  $h$  is a light hadron. We can, however, estimate the acceptable range for those terms in the same way as we did for the second term  $b$ . Results for the third and the fourth term ( $c$  and  $d$ , respectively), are shown in Figures 2 and 3. Naturally, for higher expansion parameters the uncertainty is increasing. Nevertheless, if  $k_{min}$  were close to 1,

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<sup>5</sup>We note here that perturbative corrections are expected to be at most 10-15%. For  $\mathcal{F}(\omega)$  in [14] it was found that  $\hat{a} \simeq \tilde{a} - 0.06$  and  $\hat{b} \simeq \tilde{b} - 0.06 - 0.06\tilde{a}$ , while for  $\mathcal{G}(\omega)$  corresponding results were found to be  $\hat{a} \simeq \tilde{a} + 0.02$  and  $\hat{b} \simeq \tilde{b} + 0.01 + 0.02\tilde{a}$ .



the range of acceptable values for  $c$  and  $d$  would be quite narrow. We note results for  $c$  and  $d$  obtained for the physically motivated case of  $k = 1$ , where we find

$$c \simeq -0.03 + 0.38a^2 + 0.50a^3, \quad (28)$$

$$d \simeq -0.01 - 0.03a + 0.09a^2 + 0.38a^3 + 0.31a^4. \quad (29)$$

One should also observe that all expansion parameters are roughly of the same order of magnitude, so that the only suppression for  $n$ -th order term in the expansion (4) is due to a factor of  $(\omega - 1)^n$ . This fact is best illustrated by taking  $a \simeq 1$  in (27), (28) and (29), which leads to  $b \simeq 0.94$ ,  $c \simeq 0.85$  and  $d \simeq 0.74$ . Taking these values near the maximum velocity transfer in  $B \rightarrow D^* l \bar{\nu}_l$  decays ( $\omega_{max} \simeq 1.5$ ), we find from (4)

$$\xi(1.5) \simeq 1 - 0.50 + 0.235 - 0.106 + 0.046 - \dots. \quad (30)$$

Although the curvature term is large ( $\sim 0.235$ ), the subsequent terms are not negligible, and it is obvious that series converges slowly for  $\omega \simeq \omega_{max}$ .

In order to show the effects of increasing of the number of terms in the form factor expansion about  $\omega = 1$ , in Figure 4 we show what happens as we include one, two, three, and four terms in (4), for  $k = 1$  and  $a = 1$ . Clearly, in this case keeping only the first two terms ( $a$  and  $b$  non-zero) in (4) is an excellent approximation to  $\xi(\omega)$  for  $\omega \leq 1.2$ . However, as  $\omega$  increases, the higher order terms make a difference as far as the shape of the form factor is concerned. For slope values smaller than one the form factor convergence is better. This is illustrated in Figure 5, where we have used a slope of  $a = 0.75$ . Nevertheless, in the analysis of the experimental data one has to keep in mind that for larger values of the slope it may still be important to include at least the third term in the form factor expansion about  $\omega = 1$ .

## 5 Conclusion

Within the framework of the general quark model, we have addressed the issue of the bounds on curvature and higher derivatives in the expansion of the Isgur-Wise function about the zero recoil point. These terms are important in experimental extrapolation of the form factor towards  $\omega = 1$ . Except for slopes in the range of 0.5 to 0.7, our results indicate slightly larger curvature than the one obtained by Caprini and Neubert [14]. We also find that including a third term in the form factor expansion about  $\omega = 1$  may be important in the analysis of the experimental data.

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# TABLES

Table 1: Parameters  $\beta$ ,  $\gamma$  and  $\delta$  for  $k = 2$ ,  $k = 1$  and  $k = 1/2$ .

$k$	$\beta$	$\gamma$	$\delta$
2	$\frac{5}{3}(\approx 1.67)$	$\frac{35}{9}(\approx 3.89)$	$\frac{35}{3}(\approx 11.67)$
1	$\frac{5}{2}(= 2.50)$	$\frac{35}{3}(\approx 11.67)$	$\frac{175}{2}(= 87.50)$
1/2	$\frac{715}{126}(\approx 5.67)$	$\frac{60775}{567}(\approx 107.19)$	$\frac{5773625}{1134}(\approx 5091.38)$

## FIGURES

Figure 1: The valence quark model prediction for bounds on the IW function curvature  $b$  in terms of its slope  $a$  (full lines). The line denoted by  $k_{min} = 1/2$  represents the upper, while the line denoted by  $L.B.$  represents the lower bounds. The dashed line shows the result of the CN analysis [14].

Figure 2: The valence quark model predictions for bounds on the third term  $c$  in the expansion of the IW function (4), in terms of the slope parameter  $a$ . The line denoted by  $k_{min} = 1/2$  represents the upper, while the line denoted by  $L.B.$  represents the lower bounds.

Figure 3: The valence quark model predictions for bounds on the fourth term  $d$  in the expansion of the IW function (4), in terms of the slope parameter  $a$ . The line denoted by  $k_{min} = 1/2$  represents the upper, while the line denoted by  $L.B.$  represents the lower bounds.

Figure 4: Illustration of the effects of increasing the number of terms in the form factor expansion about  $\omega = 1$ . We used  $a = 1$  and the physically motivated case of  $k = 1$ .

Figure 5: Same as in Figure 4 except with a slope parameter  $a = 0.75$ .

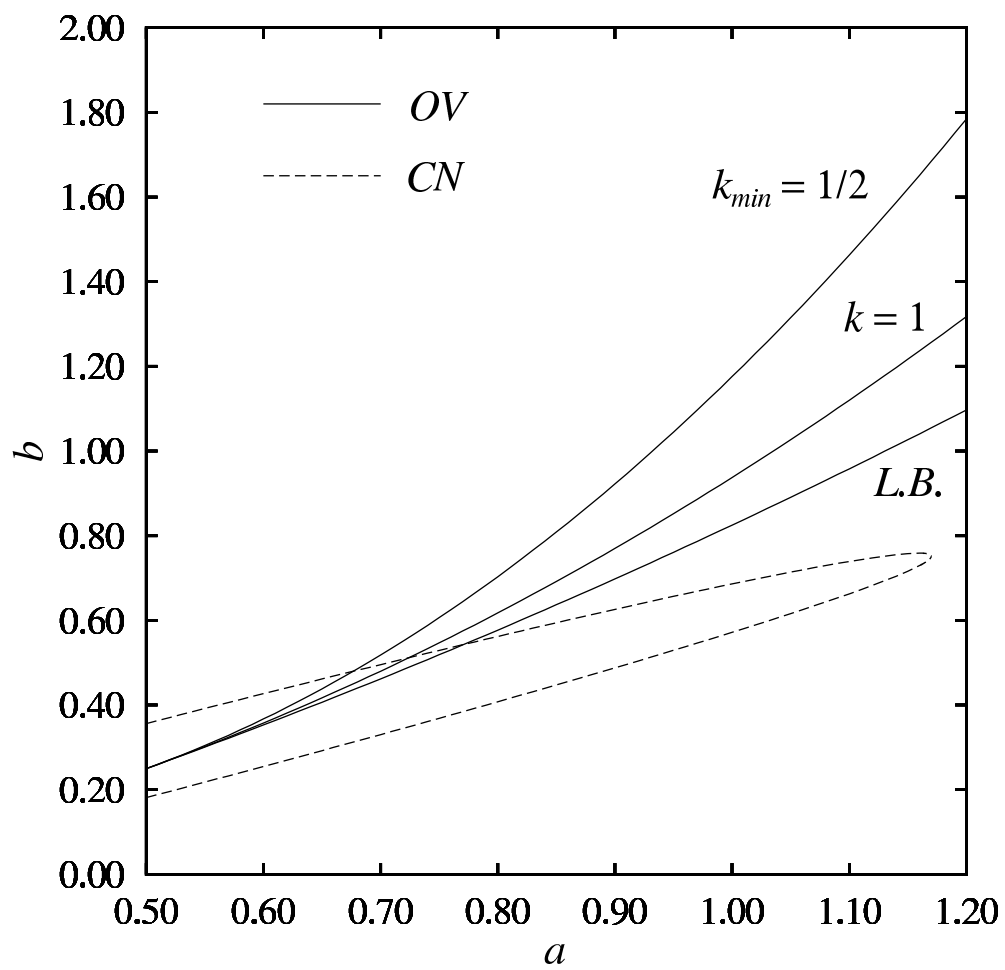


Figure 1

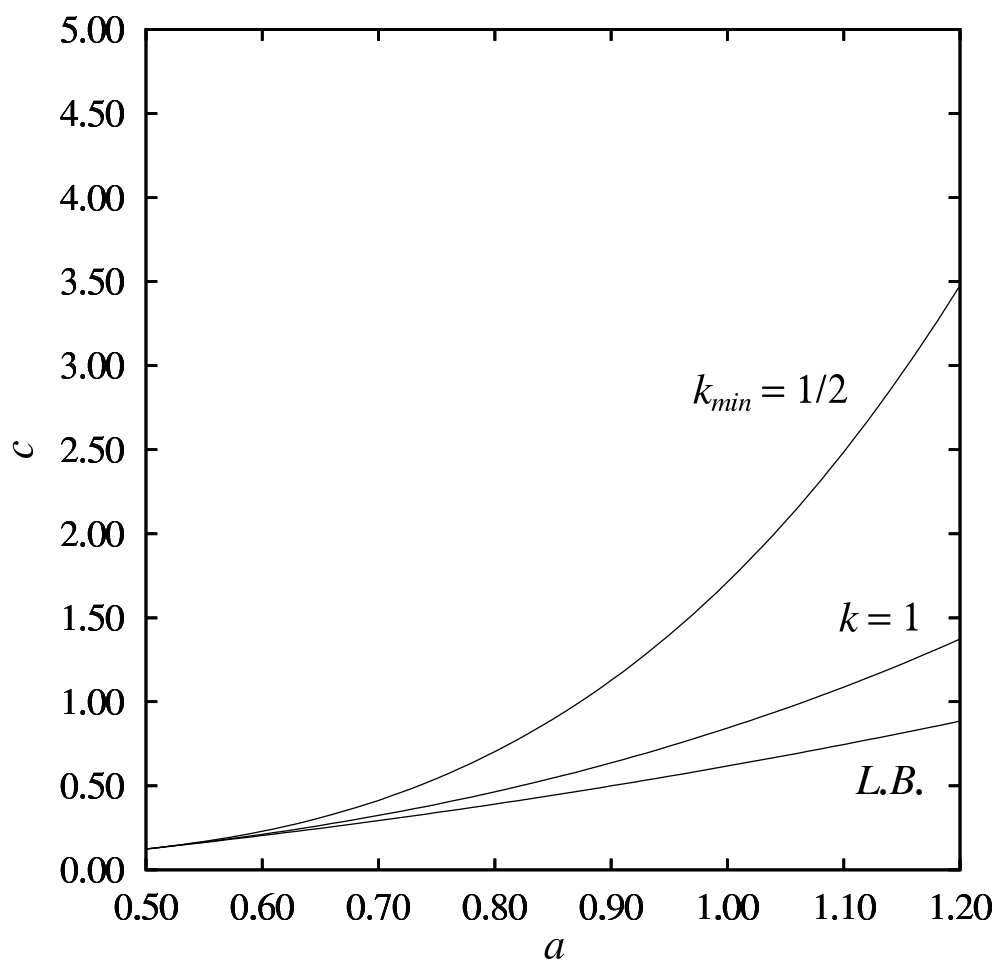


Figure 2



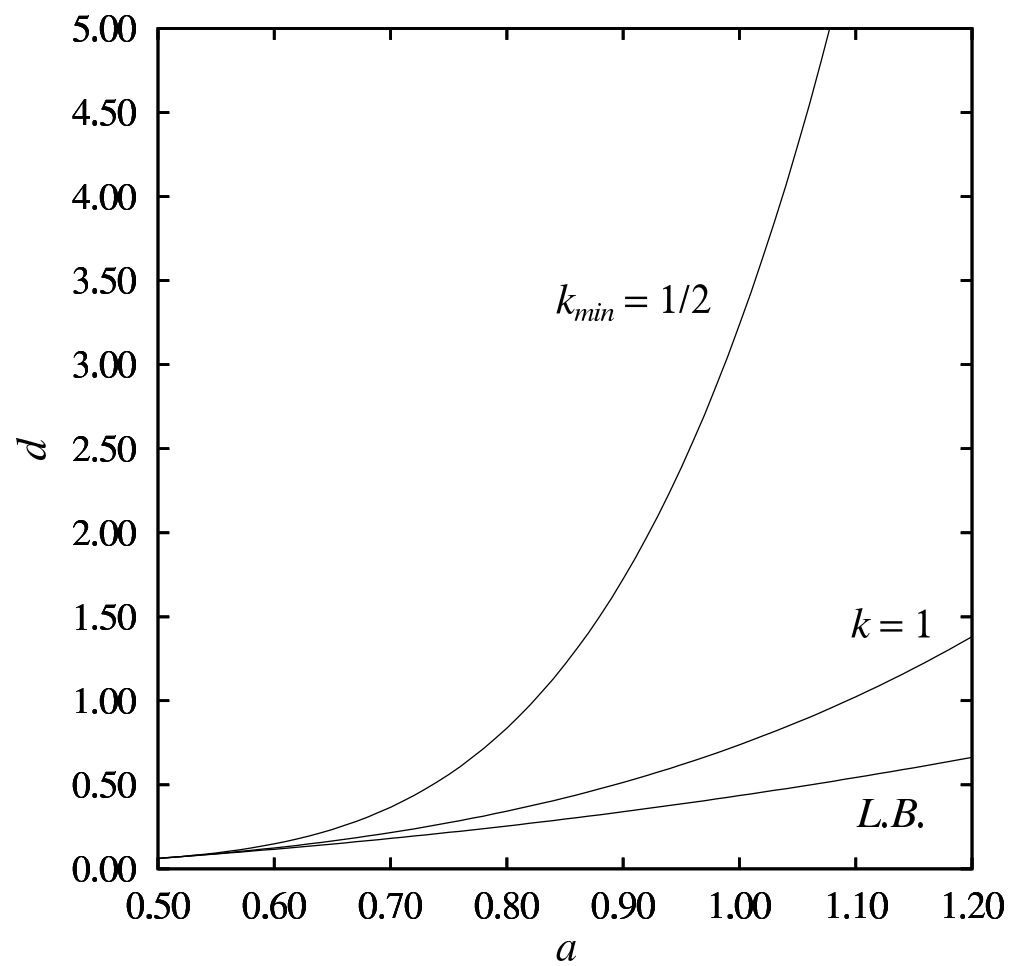


Figure 3

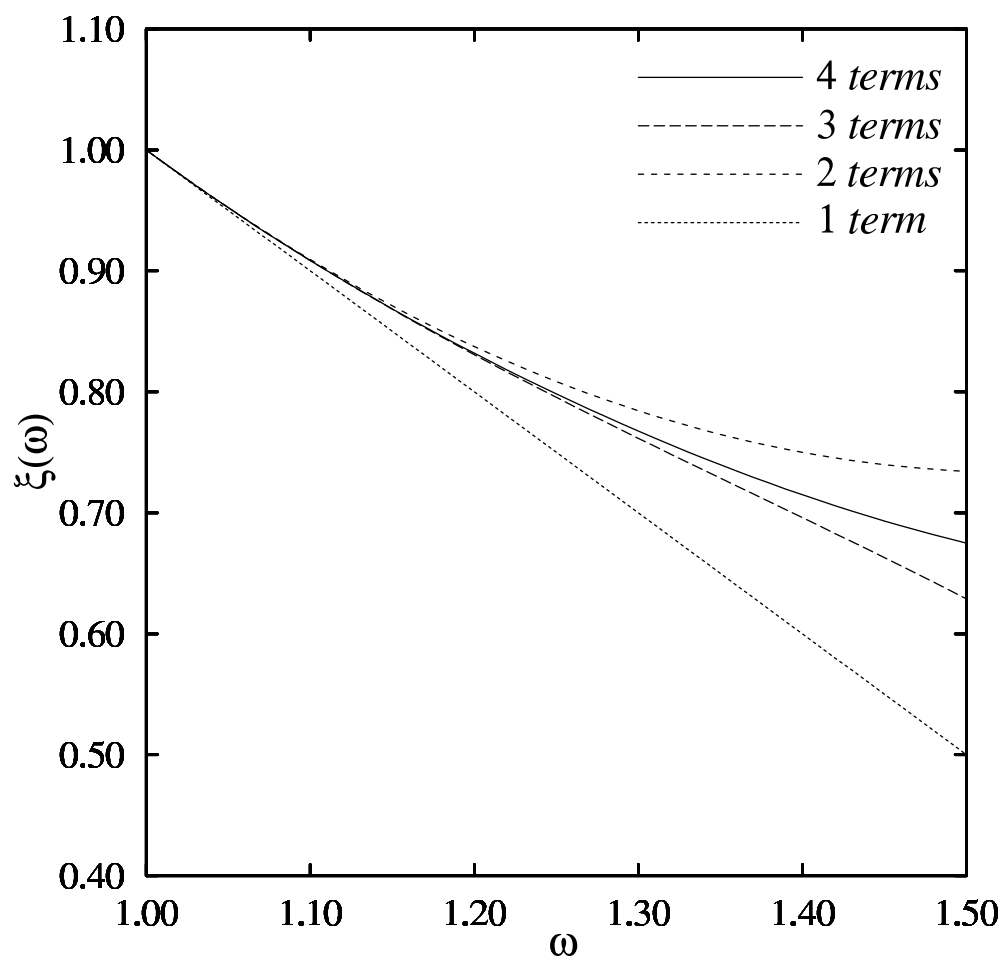


Figure 4

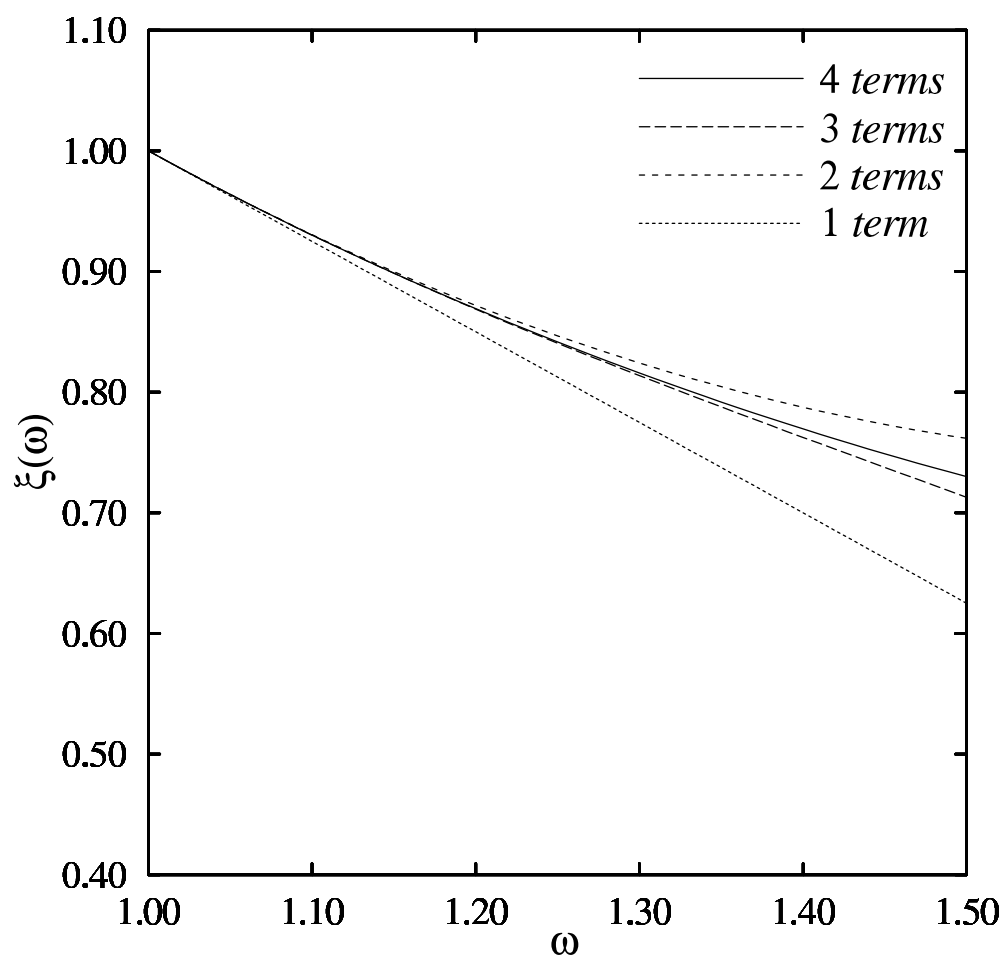


Figure 5